Completely Independent Spanning Trees in Some Regular Networks

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Abstract

Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees of a graph G. If for any pair of vertices (u, v) of $V(G)$, the paths from u to v in each T_i, $1 \leq i \leq k$, do not contain common edges and common vertices, except the vertices u and v, then $T_1, \ldots, T_k$ are completely independent spanning trees in G. For $2k$-regular graphs which are $2k$-connected, such as the Cartesian product of a complete graph of order $2k-1$ and a cycle and some Cartesian products of three cycles (for $k = 3$), the maximum number of completely independent spanning trees contained in these graphs is determined and it turns out that this maximum is not always k.

Keywords: Spanning tree, Cartesian product, Completely independent spanning tree.

1 Introduction

Let $k \geq 2$ be an integer and $T_1, \ldots, T_k$ be spanning trees in a graph G. The spanning trees $T_1, \ldots, T_k$ are *edge-disjoint* if $E(T_1) \cap \cdots \cap E(T_k) = \emptyset$. For a given tree T and a given pair of vertices (u, v) of T, let $P_T(u, v)$ be the set of vertices in the unique path between u and v in T. The spanning trees $T_1, \ldots, T_k$ are *internally disjoint* if for any pair of vertices (u, v) of $V(G)$, $P_{T_1}(u, v) \cap \cdots \cap P_{T_k}(u, v) = \{u, v\}$. Finally, the spanning trees $T_1, \ldots, T_k$ are *completely independent spanning trees* if they are pairwise edge disjoint and internally disjoint.

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Disjoint spanning trees have been extensively studied as they are of practical interest for fault-tolerant broadcasting or load-balancing communication systems in interconnection networks : a spanning-tree is often used in various network operations; computing completely independent spanning-trees guarantees a continuity of service, as each can be immediately used as backup spanning tree if a node or link failure occurs on the current spanning tree. Thus, computing k completely independent spanning trees allows to handle up to $k-1$ simultaneous independent node or link failures. In this context, a network is often modeled by a graph G in which the set of vertices $V(G)$ corresponds to the nodes set and the set of edges $E(G)$ to the set of direct links between nodes.

Completely independent spanning trees were introduced by T. Hasunuma [4] and then have been studied on different classes of graphs, such as underlying graphs of line graphs [4], maximal planar graphs [6], Cartesian product of two cycles [7] and complete graphs, complete bipartite and tripartite graphs [11]. Moreover, the decision problem that consists in determining if there exist two completely independent spanning trees in a graph G is NP-hard [6].

Other works on disjoint spanning trees include independent spanning trees which focus on finding spanning trees $T_1, \dots, T_k$ rooted at r, such that for any vertex v the paths from r to v in $T_1, \dots, T_k$ are pairwise openly disjoint. the main difference is that $T_1, \dots, T_k$ are rooted at r and only the paths to r are considered. Thus $T_1, \dots, T_k$ may share common edges, which is not admissible with completely independent spanning trees. Independent spanning trees have been studied in several topologies, including product graphs [10], de Bruijn and Kautz digraphs [3, 5], and chordal rings [9]. Related works also include Edge-disjoint spanning trees, i.e. spanning-trees which are pairwise edge disjoint only. Edge-disjoint spanning trees have been studied on many classes of graphs, including hypercubes [1], Cartesian product of cycles [2] and Cartesian product of two graphs [8].

We use the following notations : for a tree, a vertex that is not a leaf is called an *inner vertex*. For a vertex u of a graph G, let $d_G(u)$ be its degree in G, i.e. the number of edges of G incident with it.

For clarity, we recall the definition of the Cartesian product of two graphs : Given two graphs G and H, the Cartesian product of G and H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,u')(v,v') | (u = v \wedge u'v' \in E(H)) \vee (u' = v' \wedge uv \in E(G))\}$.

The following theorem gives an alternative definition [4] of completely independent spanning trees.

Theorem 1.1 ([4])**.** *Let $k \geq 2$ be an integer. $T_1, \dots, T_k$ are completely independent spanning trees in a graph G if and only if they are edge-disjoint spanning trees of G and for any $v \in V(G)$, there is at most one T_i such that $d_{T_i}(v) > 1$.*

It has been conjectured that in any $2k$-connected graph, there are k completely independent spanning trees [6]. This conjecture has been refuted, as there exist $2k$-connected graphs which do not contain two completely independent spanning trees [12], for any integer k. However, the given counterexamples are not $2k$-regular.

Proposition 1.2 ([12]). *For any $k \geq 2$, there exist $2k$-connected graphs that do not contain two completely independent spanning trees.*

The proof of the previous proposition consists in constructing a $2k$-connected graph with a large proportion of vertices of degree $2k$ adjacent to the same vertices and proving that these vertices of degree $2k$ can not be all adjacent to inner vertices in a fixed tree.

This article is organized as follows. Section 2 presents necessary conditions on $2r$-regular graphs in order to have r completely independent spanning trees. Section 3 presents the maximum number of completely independent spanning trees in $K_m \square C_n$, for $n \geq 3$ and $m \geq 3$. In particular, we exhibit the first $2r$-regular graphs which are $2r$-connected and which do not contain r completely independent spanning trees. In Section 4, we determine three completely independent spanning trees in some Cartesian products of three cycles $C_{n_1} \square C_{n_2} \square C_{n_3}$, for $3 \leq n_1 \leq n_2 \leq n_3$.

2 Necessary conditions on $2r$-regular graphs

Proposition 2.1. *If in a $2r$-regular graph G there exist r completely independent spanning trees, then every spanning tree has maximum degree at most $r+1$.*

Proof. By Theorem 1.1, every vertex should be of degree 1 in every spanning tree except in one spanning tree. Hence, in a spanning tree, a vertex is either of degree 1 (a leaf) or has degree between 2 and $r+1$ (an inner vertex), as $2r - (r-1) = r+1$. $\square$

Let $\mathrm{IN}(T)$ be the set of inner vertices in a tree T.

Proposition 2.2. *If in a $2r$-regular graph G of order n there exist r completely independent spanning trees, then there exists a spanning tree T among them such that $|IN(T)| \leq \lfloor n/r \rfloor$.*

Proof. Let $T_1, \ldots, T_r$ be completely independent spanning trees in G and suppose that $|\mathrm{IN}(T_i)| > \lfloor n/r \rfloor$ for every $i \in \{1, \ldots, r\}$. By Theorem 1.1, we have $\sum_{i=1}^{r} |\mathrm{IN}(T_i)| \leq n$. With our hypothesis, we have $\sum_{i=1}^{r} |\mathrm{IN}(T_i)| \geq (\lfloor n/r \rfloor + 1)r > n$, and a contradiction. $\square$

Proposition 2.3. *If in a $2r$-regular graph G of order n there exist r completely independent spanning trees $T_1, \ldots, T_r$, then for every integer i, $1 \leq i \leq r$,*

$$\left\lceil \frac{n-2}{r} \right\rceil \leq |IN(T_i)| \leq n - \left\lceil \frac{n-2}{r} \right\rceil (r-1).$$

Proof. In a spanning tree T of a graph of order n we recall that $\sum_{v \in V(T)} d_T(v) = 2n-2$. By Proposition 2.1, we have $\sum_{v \in V(T)} d_T(v) \leq |\mathrm{IN}(T)|r + n$ and we obtain $\lceil \frac{n-2}{r} \rceil \leq |\mathrm{IN}(T)|$. By Theorem 1.1, $\sum_{i=1}^{r} |\mathrm{IN}(T_i)| \leq n$. For a fixed integer i, using the previous inequality, we obtain $|\mathrm{IN}(T_i)| \leq n - \lceil \frac{n-2}{r} \rceil (r-1)$. $\square$

Definition 2.1. *Let G be a $2r$-regular graph of order n for which there exist r completely independent spanning trees $T_1, \ldots, T_r$. A* lost edge *is an edge of G that is in none of the spanning trees $T_1, \ldots, T_r$. We let E^l be the set of lost edges, i.e. $E^l = E(G) - \bigcup_{1 \leq i \leq r} E(T_i)$. Let also $E^l_{T_i} = \{uv \in E(G) | u, v \in IN(T_i),\ uv \notin E(T_i)\}$, for $i \in \{1, \ldots, r\}$, i.e. $E^l_{T_i}$ is the subset of edges of E^l that have their two extremities in $IN(T_i)$.*

Proposition 2.4. *If in a $2r$-regular graph G of order n there exist r completely independent spanning trees $T_1, \ldots, T_r$, then $|E^l| = r$.*

Proof. We have $\sum_{i=1}^{r} |E(T_i)| + |E^l| = E(G) = rn$ and $\sum_{i=1}^{r} |E(T_i)| = r(n-1)$. Hence, $|E^l| = r$. $\square$

Since each edge of $E^l_{T_i}$ is also in E^l and each edge of E^l is in at most one set $E^l_{T_i}$ for some integer i, we have the following observation.

Observation 2.5. *In a $2r$-regular graph G of order n for which there exist r completely independent spanning trees $T_1, \ldots, T_r$, we have $\sum_{1 \leq i \leq r} |E^l_{T_i}| \leq |E^l| = r$.*

Definition 2.2. *The* potential extra degree *of a spanning tree T in a $2r$-regular graph G of order n is $ped(T) = |IN(T)|r - n + 2$.*

With Proposition 2.3, we have the following easy observation:

Observation 2.6. *Let G be a graph, for which there exist r completely independent spanning trees $T_1, \ldots, T_r$. Then, for every i, $0 \leq i \leq r$, $ped(T_i) \geq 0$.*

Note also that, by definition, the number of inner vertices of T_i of degree at most r is bounded by $\mathrm{ped}(T_i)$.

Proposition 2.7. *If in a $2r$-regular graph G of order n there exist r completely independent spanning trees, then there exists a spanning tree T among them such that $ped(T) \leq 2$ and $E^l_T \leq 1$, with strict inequalities if r does not divide n.*

Proof. By Proposition 2.2, there exists a tree T among them such that $|\mathrm{IN}(T)| \leq \lfloor n/r \rfloor$. Hence, $\mathrm{ped}(T) \leq \lfloor n/r \rfloor r - n + 2 \leq 2$, with strict inequality if r does not divide n. For every edge uv in E^l_T, both u and v are adjacent to one inner vertex of every spanning tree other than T. Hence, both u and v have degree at most r in T and thus $\mathrm{ped}(T) \geq 2|E^l_T|$. $\square$

Note that the inequality $\mathrm{ped}(T) \geq 2|E_T^l|$ can be strict.

Corollary 2.8. *Suppose that G is a $2r$-regular graph of order n for which there exist r completely independent spanning trees $T_1, \ldots, T_r$, for $r \geq 3$ and $n \equiv 0 \pmod r$. Then, for every integer i, $1 \leq i \leq r$, $|IN(T_i)| = n/r$ and $ped(T_i) = 2$.*

Observation 2.9. *For a $2r$-regular graph G of order n for which there exist r completely independent spanning trees $T_1, \ldots, T_r$, for every tree T_i, $1 \leq i \leq r$, and every edge e in $E_{T_i}^l$, the extremities of e have degree at most r in T_i.*

3 Cartesian product of a complete graph and a cycle

Let $m \geq 3$ and $n \geq 2$ be integers. In this section, the considered graphs are $K_m \Box P_n$, and $K_m \Box C_n$ $n \geq 3$.

Let $V(K_m \Box P_n) = V(K_m \Box C_n) = \{u_i^j,\ 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ and $E(K_m \Box P_n) = \{u_i^j u_k^j,\ 0 \leq i, k \leq m-1, i \neq k, 0 \leq j \leq n-1\} \cup \{u_i^j u_i^{j+1},\ 0 \leq i \leq m-1, 0 \leq j \leq n-2\})$. $E(K_m \Box C_n) = E(K_m \Box P_n) \cup \{u_i^0 u_i^{n-1},\ 0 \leq i \leq m-1\}$.

For $j \in \{0, \ldots, n-1\}$, the subgraphs K^j induced by $\{u_i^j,\ 0 \leq i \leq m-1$ are thus complete graphs on n vertices that we call *K-copies.* In order to study the distribution of inner vertices of the spanning trees among the K-copies, we let $V_j(T) = \mathrm{IN}(T) \cap V(K^j)$ and $n_j(T) = |V_j(T)|$ for any spanning tree T of $K_m \Box C_n$.

In the remaining, the subscript of u_i^j is considered modulo m and its superscript and the subscripts of $V_j(T)$ and $n_j(T)$ are considered modulo n.

Proposition 3.1. *Let n and r be integers, $n \geq 2$, $r \geq 2$. There exist r completely independent spanning trees in $K_{2r} \Box P_n$.*

Proof. We construct r completely independent spanning trees $T_1, \ldots, T_r$ as follows: $E(T_i) = \{u_{i-1}^j u_{i-1}^{j+1}, u_{r+i-1}^j u_{r+i-1}^{j+1} | j \in \{0, \ldots, n-2\}\} \cup \{u_{i-1}^0 u_{r+i-1}^0\} \cup \{u_{i-1}^j u_{i+k}^j, u_{r+i-1}^j u_{r+i+k}^j | k \in \{0, \ldots, r-2\}\ , j \in \{0, \ldots, n-1\}\}$. □

Corollary 3.2. *Let n and r be integers, $n \geq 3$, $r \geq 2$. There exist r completely independent spanning trees in $K_{2r} \Box C_n$.*

In the three next propositions, we will prove that there do not exist r completely independent spanning trees in $K_{2r-1} \Box C_n$, for some integers r and n. Let $p = |V(K_{2r-1} \Box C_n)| = n(2r-1)$ and assume that there exist r completely independent spanning trees $T_1, \ldots, T_r$ in $K_{2r-1} \Box C_n$. Let T be the spanning tree among them which minimizes $|\mathrm{IN}(T)|$, i.e. $\mathrm{ped}(T)$. By Proposition 2.2, T is such that $|\mathrm{IN}(T)| \leq \lfloor p/r \rfloor = 2n - \lceil n/r \rceil$, $\mathrm{ped}(T) \leq 2nr - \lceil n/r \rceil r - p + 2 \leq n - \lceil n/r \rceil r + 2 \leq 2$ and $|E_T^l| \leq 1$. In order to establish this property we will consider all possible distributions of inner vertices of T among the different K-copies and prove that for each of them we have a contradiction.

The properties given in the following lemma will be useful.

Lemma 3.3. *Let $a_i(T)$ be the number of K-copies which contains exactly i inner vertices of T. The distribution of inner vertices among the different K-copies is such that:*

i) if $n_j(T) \geq k$, for some integer j, then $|E^l_T| \geq \frac{1}{2}(k-1)(k-2)$;

ii) $n_j(T) < 4$, for every integer j;

iii) $a_3(T) \leq 1$;

iv) if $a_3(T) = 1$, then $n \equiv 0 \pmod r$ and $n \geq r$;

v) if $a_0(T) = 0$, then $a_3(T) \leq a_1(T) - \lceil n/r \rceil$; in particular $a_1(T) > a_3(T)$ and $a_1(T) \geq \lceil n/r \rceil$.

Proof. i) : A complete graph of order k contains $\frac{1}{2}k(k-1)$ edges and only $k-1$ edges are in $E(T)$. Thus we have $|E^l_T| \geq \frac{1}{2}(k-1)(k-2)$.
ii) and iii) : If $n_j(T) \geq 4$ for some j or $a_3(T) > 1$, then by i), we have $|E^l_T| \geq 2$. Hence, a contradiction.
iv) : As $\mathrm{ped}(T) \leq n - \lceil n/r \rceil r + 2$, we have $|E^l_T| < 1$ in the case $n \not\equiv 0 \pmod r$. As $n > 0$, we have $n \geq r$.
v) : By ii), we have $|\mathrm{IN}(T)| = a_1(T) + 2a_2(T) + 3a_3(T)$ and $a_2 = n - a_1(T) - a_3(T)$. Hence $|\mathrm{IN}(T)| = a_1(T) + 2(n - a_1(T) - a_3(T)) + 3a_3(T) \leq 2n - \lceil n/r \rceil$ by the choice of T. Thus, $a_3(T) \leq a_1(T) - \lceil n/r \rceil$ and consequently $a_1(T) > a_3$ and $a_1(T) \geq \lceil n/r \rceil$. □

We recall the following observation used in [12].

Observation 3.4 ([12]). *If in a graph G there exist r completely independent spanning trees $T_1, \ldots, T_r$, then for every integer i, $1 \leq i \leq r$, every vertex is adjacent to an inner vertex of T_i.*

Proposition 3.5. *Let n, r be integers, with $n \geq 3$ and $r \geq 6$. There do not exist r completely independent spanning trees in $K_{2r-1} \square C_n$.*

Proof. The proof is by contradiction, using Properties i)-v) of Lemma 3.3. Suppose that there exist r completely independent spanning trees in $K_{2r-1} \square C_n$ and let T be the tree from Proposition 2.2. If a K-copy K^i, $1 \leq i \leq n$, contains no inner vertex, then, by Observation 3.4, $n_{i-1}(T) + n_{i+1}(T) \geq 2r - 1 \geq 11$. Consequently, we have $n_{i-1}(T) \geq 6$ or $n_{i+1}(T) \geq 6$, contradicting Property ii). Hence $a_0(T) = 0$.

By Property v), $a_1(T) \geq \lceil n/r \rceil \geq 1$. Hence there exists an integer i, $0 \leq i \leq n-1$, such that $n_i = 1$. Let u be the (unique) vertex of $V_i(T)$. The vertex u has degree at most $r+1$ in T and is adjacent in T to a vertex of $V_{i-1}(T) \cup V_{i+1}(T)$. Then, u is adjacent in T to at most r vertices of $V(K^i)$. Thus, at least $r - 2 \geq 4$ vertices are not adjacent in T to u. Hence, these $r-2$ vertices are adjacent in T to vertices of $V_{i-1}(T) \cup V_{i+1}(T)$ and consequently $n_{i-1}(T) + n_{i+1}(T) \geq 5$. Therefore, we have $n_{i-1}(T) \geq 3$ or $n_{i+1}(T) \geq 3$.

Assume, without loss of generality, that $n_{i+1}(T) \geq 3$. By Property ii), $n_{i+1}(T) = 3$ and by Property iii), $a_3(T) = 1$, i.e., $n_j(T) < 3$ for any $j \neq i$.

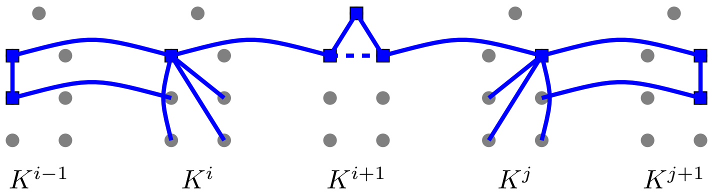

Figure 1: A configuration of inner vertices in the proof of Proposition 3.6. Boxes are inner vertices and the dashed edge represents a lost edge.

But, by Property iv), $n \geq r$ and by Property v), $a_1 \geq 2$. Let j be such that $n_j(T) = 1$, with $j \neq i$. Using a similar argument than above, we obtain that $n_{j-1}(T) \geq 3$ or $n_{j+1}(T) \geq 3$. But, as $a_3(T) = 1$, the only possibility is to have $j = i+2$, i.e. both K-copies with one internal vertices are adjacent to the same K-copy with three internal vertices.

Let v be the (unique) vertex of $V_j(T)$. One vertex among u and v is adjacent in T to two inner vertices (if not T would be not connected). Suppose, without loss of generality, that u is adjacent in T to two inner vertices. Then u is adjacent in T to at most $r-1$ vertices in $V(K^i)$. Thus, at least $r-1 \geq 5$ vertices are not adjacent in T to u. Therefore, at least 5 vertices are adjacent in T to vertices of $V_{i-1}(T) \cup V_{i+1}(T)$ and consequently $n_{i-1}(T) + n_{i+1}(T) \geq 7$. Hence, we have $n_{i-1}(T) \geq 4$ or $n_{i+1}(T) \geq 4$, contradicting Property ii). $\square$

Proposition 3.6. *Let n, r be integers, with $4 \leq r \leq 5$ and $n \geq r+1$. There do not exist r completely independent spanning trees in $K_{2r-1} \Box C_n$.*

Proof. The proof is by contradiction, using Properties i)-v) of Lemma 3.3. Suppose that there exist r completely independent spanning trees in $K_{2r-1} \Box C_n$ and let T be the tree from Proposition 2.2. If a K-copy K^i, $0 \leq i \leq n-1$, contains no inner vertex, then $n_{i-1}(T) + n_{i+1}(T) \geq 7$. Consequently, we have $n_{i-1}(T) \geq 4$ or $n_{i+1}(T) \geq 4$, contradicting Property ii). Hence $a_0(T) = 0$. By Property v), $a_1(T) \geq \lceil n/r \rceil \geq 2$. Thus, there exist two integers i and j, $0 \leq i \leq j \leq n-1$, such that $n_i(T) = n_j(T) = 1$, with $u \in V_i(T)$ and $v \in V_j(T)$.

First, suppose that $i = j-1$. Each of u and v has degree at most $r+1$ in T and u (v, respectively) is adjacent in T to a vertex of $V_{i-1}(T) \cup V_{i+1}(T)$ (of $V_{j-1}(T) \cup V_{j+1}(T)$, respectively).

If u and v are adjacent in T, then one vertex among u and v is adjacent in T to a vertex of $V_{i-1}(T) \cup V_{j+1}(T)$ (if not T would be not connected). Suppose, without loss of generality, that u is adjacent to two inner vertices. Then, at least $r-1 \geq 3$ vertices of $V(K^i)$ are not adjacent in T to u. Consequently, $n_{i-1}(T) \geq 4$ and we have a contradiction with Property ii).

Else if u and v are not adjacent in T, then both u and v are adjacent in T to vertices of $V_{i-1}(T) \cup V_{j+1}(T)$ (if not, T would be not connected). The vertices u and v are each adjacent in T to at most r vertices in $V(K^i) \cup V(K^j)$. Hence, there remain at least $4r - 2 - 2r - 2 = 2r - 4 \geq 4$ vertices in $V(K^i) \cup V(K^j)$ that must be adjacent in T to vertices of $V_{i-1}(T) \cup V_{j+1}(T)$ other than the neighbors of u and of v. Consequently $n_{i-1}(T) + n_{j+1}(T) \geq 6$. Hence, we have $n_{i-1}(T) \geq 3$ and $n_{j+1}(T) \geq 3$, contradicting Property iii) or $n_{i-1}(T) \geq 4$ or $n_{j+1}(T) \geq 4$, contradicting Property ii).

Second, if $|i - j| > 1$, then one vertex among u and v is adjacent in T to two inner vertices (if not T would be not connected). Suppose, without loss of generality, that u is adjacent to two inner vertices. At least $r - 1$ vertices of $V(K^i)$ are not adjacent in T to u. Hence, if $r = 5$, we have $n_{i-1}(T) \geq 3$ and $n_{i+1}(T) \geq 3$, contradicting Property iii) or $n_{i-1}(T) \geq 4$ or $n_{i+1}(T) \geq 4$, contradicting Property ii). Consequently, we suppose that $r = 4$. Then, at least $r - 1 \geq 3$ vertices of $V(K^i)$ are not adjacent in T to u. Therefore, we have $n_{i-1}(T) \geq 3$ or $n_{i+1}(T) \geq 3$.

Assume, without loss of generality, that $n_{i+1}(T) \geq 3$. By Property ii), $n_{i+1}(T) = 3$ and by Property iii), $a_3(T) = 1$, i.e., $n_j(T) < 3$ for any $j \neq i$. But, as $n > r$ and by Property v), $a_1 \geq 3$. Let i' be such that $n_{i'}(T) = 1$, with $i' \neq i$ and $i' \neq i$. If $|i' - i| = 1$ or $|i' - j| = 1$, we have a contradiction, using the first point. Two vertices among u, v and u' should be adjacent to two inner vertices. Suppose it is the vertices u and v. Using a similar argument than above, we obtain that $n_{j-1}(T) \geq 3$ or $n_{j+1}(T) \geq 3$. But, as $a_3(T) = 1$, the only possibility is to have $j = i + 2$, i.e. both K-copies with one internal vertices are adjacent to the same K-copy with three internal vertices.

In this case, as $r = 4$, then four vertices are not inner vertices in $V(K^{i+1})$, at least three vertices of $V(K^i)$ are not adjacent in T to u and at least three vertices of $V(K^j)$ are not adjacent in T to v. Moreover, we have $n_{i-1}(T) \leq 2$ and $n_{j+1}(T) \leq 2$. Figure 1 illustrates this configuration. Thus, four vertices of $V(K^{i+1})$ are adjacent in T to vertices of $V_{i+1}(T)$ and four vertices of $V(K^i) \cup V(K^j)$ are adjacent in T to vertices of $V_{i+1}(T)$. However, by Observation 2.9, the vertices of $V_{i+1}(T)$ can be adjacent to at most seven leaves in T. Hence, we have a contradiction.

$\square$

Proposition 3.7. *There do not exist five completely independent spanning trees in $K_9 \square C_3$.*

Proof. Suppose that there exist five completely independent spanning trees in $K_9 \square C_3$ and let T be the tree from Proposition 2.2. We recall that $|V(K_9 \square C_3)| = 27$ and $|\mathrm{IN}(T)| \leq 6 - \lceil 3/4 \rceil = 5$. If a K-copy K^i, $0 \leq i \leq n-1$, contains no inner vertex, then $n_{i-1}(T) \geq 5$ or $n_{i+1}(T) \geq 5$. Thus, we have a contradiction with Property ii). By property iv), as $n \not\equiv 0 \pmod r$, we have $a_3(T) = 0$. Thus, the only possible distribution of inner vertices of T is $a_1(T) = 1$ and $a_2(T) = 2$. Without loss of generality, suppose that $n_0(T) = 1$, $n_1(T) = 2$ and $n_2(T) = 2$, with $u \in V_1(T)$.

Let the position of a vertex u_i^j be i. As T should be connected, two pairs of inner vertices in different K-copies should be adjacent in T among these five inner vertices. Thus, these five vertices have only three different positions. The

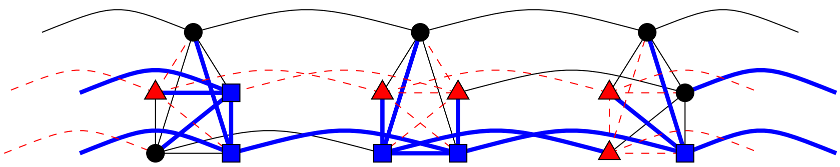

Figure 2: A pattern to have three completely independent spanning trees in $K_5 \square C_n$, for $n \equiv 0 \pmod 3$.

vertex u has degree at most 6 in T. Hence, there are $r-2 \geq 3$ vertices of $V(K^1)$ not adjacent in T to u. As the inner vertices have only two positions different from the position of u, it is impossible that every vertex is adjacent in T to an inner vertex of T. $\square$

We now show positive results for the remaining values of r and n. Some of the spanning trees were found using a computer to solve an ILP formulation of the problem.

Proposition 3.8. *Let $n \geq 3$ be an integer such that $n \equiv 0 \pmod 3$. There exist three completely independent spanning trees in $K_5 \square C_n$.*

Proof. We construct three completely independent spanning trees T_1, T_2 and T_3 using repeatedly the pattern illustrated in Figure 2 on each three consecutive K-copies:
$E(T_1) = \{u_0^{3j}u_0^{1+3j}, u_0^{1+3j}u_0^{2+3j}, u_0^{2+3j}u_0^{3+3j}, u_0^{3j}u_2^{3j}, u_0^{3j}u_3^{3j},$
$u_3^{3j}u_1^{3j}, u_3^{3j}u_4^{3j}, u_3^{3j}u_3^{1+3j}, u_0^{1+3j}u_1^{1+3j}, u_0^{1+3j}u_4^{1+3j}, u_2^{1+3j}u_2^{2+3j},$
$u_0^{2+3j}u_2^{2+3j}, u_0^{2+3j}u_1^{2+3j}, u_2^{2+3j}u_3^{2+3j}, u_2^{2+3j}u_4^{2+3j} | j \in \{0, \ldots, n/3-1\}\} - \{u_0^0, u_0^1\};$
$E(T_2) = \{u_1^{3j}u_1^{1+3j}, u_1^{1+3j}u_1^{2+3j}, u_1^{2+3j}u_1^{3+3j}, u_1^{3j}u_0^{3j}, u_1^{3j}u_4^{3j},$
$u_2^{3j}u_2^{1+3j}, u_1^{1+3j}u_2^{1+3j}, u_1^{1+3j}u_4^{1+3j}, u_2^{1+3j}u_0^{1+3j}, u_2^{1+3j}u_3^{1+3j}, u_1^{2+3j}u_3^{2+3j},$
$u_1^{2+3j}u_2^{2+3j}, u_3^{2+3j}u_0^{2+3j}, u_3^{2+3j}u_4^{2+3j}, u_3^{2+3j}u_3^{3+3j} | j \in \{0, \ldots, n/3-1\}\} - \{u_1^0, u_1^1\};$
$E(T_3) = \{u_4^{3j}u_4^{1+3j}, u_4^{1+3j}u_4^{2+3j}, u_4^{2+3j}u_4^{3+3j}, u_2^{3j}u_4^{3j}, u_2^{3j}u_1^{3j},$
$u_2^{3j}u_3^{3j}, u_4^{3j}u_0^{3j}, u_3^{1+3j}u_4^{1+3j}, u_3^{1+3j}u_0^{1+3j}, u_3^{1+3j}u_1^{1+3j}, u_4^{1+3j}u_2^{1+3j},$
$u_3^{1+3j}u_3^{2+3j}, u_4^{2+3j}u_0^{2+3j}, u_4^{2+3j}u_1^{2+3j}, u_2^{2+3j}u_2^{3+3j} | j \in \{0, \ldots, n/3-1\}\} - \{u_4^0, u_4^1\}.$
$\square$

Proposition 3.9. *Let $n \geq 3$ be an integer. There exist three completely independent spanning trees in $K_5 \square C_n$.*

Proof. By Proposition 3.8, there exist three completely independent spanning trees in $K_5 \square C_n$, for $n \equiv 0 \pmod 3$. For $n \equiv 1 \pmod 3$, we use the pattern from Proposition 3.8 for $K^4 \cup \ldots \cup K^{n-1}$, completed by the pieces of three completely independent spanning trees of $K^0 \cup K^1 \cup K^2 \cup K^3$ depicted in Figure 3 and whose edge sets are given in Appendix A.1. For $n \equiv 2 \pmod 3$, we use the pattern from Proposition 3.8 for $K^5 \cup \ldots \cup K^{n-1}$, completed by the pieces of three

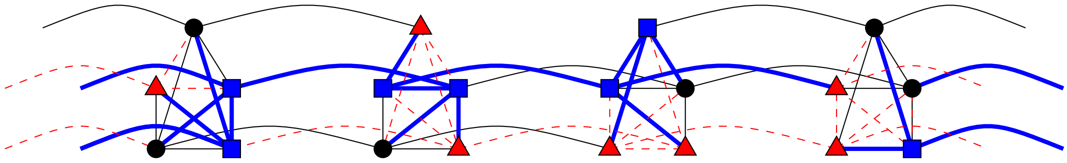

Figure 3: The three completely independent spanning trees in $K_5 \square C_n$, for $K^0 \cup K^1 \cup K^2 \cup K^3$ and $n \equiv 1 \pmod 3$.

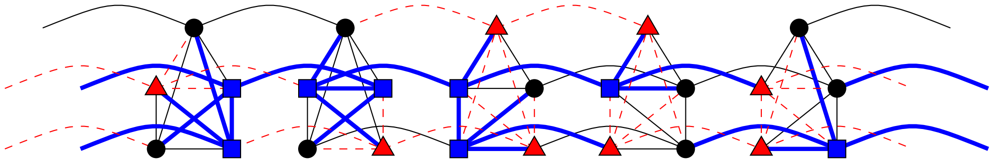

Figure 4: The three completely independent spanning trees in $K_5 \square C_n$, for $K^0 \cup K^1 \cup K^2 \cup K^3 \cup K^4$ and for $n \equiv 2 \pmod 3$.

completely independent spanning trees of $K^0 \cup K^1 \cup K^2 \cup K^3 \cup K^4$ depicted in Figure 4 and whose edge sets are given in Appendix A.2. Note that Figures 3 and 4 depicte also three completely independent spanning trees in $K_5 \square C_4$ and $K_5 \square C_5$. □

Proposition 3.10. *There exist four completely independent spanning trees in* $K_7 \square C_3$.

Proof. The four completely independent spanning trees in $K_7 \square C_3$ are depicted in Figure 5 and their edge sets are given in Appendix A.3. □

Proposition 3.11. *There exist four completely independent spanning trees in* $K_7 \square C_4$.

Proof. The four completely independent spanning trees in $K_7 \square C_4$ are depicted in Figure 6 and their edge sets are given in Appendix A.4. □

Proposition 3.12. *There exist five completely independent spanning trees in* $K_9 \square C_4$.

Proof. The five completely independent spanning trees in $K_9 \square C_4$ are depicted in Figure 7 and their edge sets are given in Appendix A.5. □

Proposition 3.13. *There exist five completely independent spanning trees in* $K_9 \square C_5$.

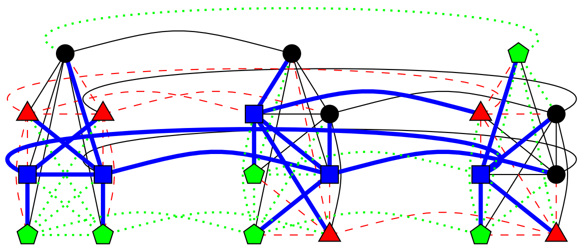

Figure 5: Four completely independent spanning trees in $K_7 \Box C_3$.

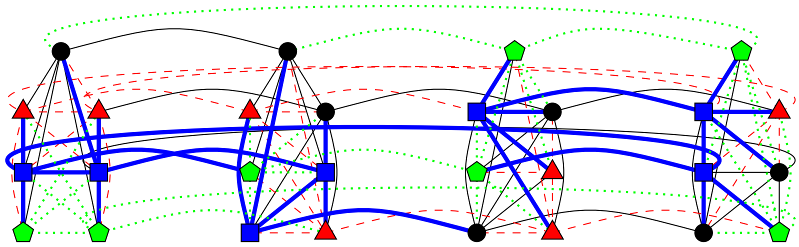

Figure 6: Four completely independent spanning trees in $K_7 \Box C_4$.

Proof. The five completely independent spanning trees in $K_9 \Box C_5$ are depicted in Figure 8 and their edge sets are given in Appendix A.6. $\square$

We end this section with a theorem summarizing the results for $K_m \Box C_n$. Given a graph G, let mcist(G) be the maximum integer k such that there exist k completely independent spanning trees in G.

Theorem 3.14. *Let $m \geq 3$ and $n \geq 3$ be integers. We have:*

$$mcist(K_m \Box C_n) = \begin{cases} \lceil m/2 \rceil, & \text{if } (m = 3, 5 \vee (m = 7 \wedge n = 3, 4) \vee (m = 9 \wedge n = 4, 5)); \\ \lfloor m/2 \rfloor, & \text{otherwise.} \end{cases}$$

Proof. For every even m, by Corollary 3.2, there exist $m/2$ completely independent spanning trees. Suppose m is odd. For $m = 3$, Hasunuma and Morisaka [7] has proven that in any Cartesian product of 2-connected graphs, there are two completely independent spanning trees. By Propositions 3.12, 3.13, 3.10, 3.11 and 3.9, we obtain that there exist $\lceil m/2 \rceil$ completely independent spanning trees for $m = 5$ or $(m = 7 \wedge n = 3, 4)$ or $(m = 9 \wedge n = 4, 5)$.

In the other cases, by Propositions 3.5, 3.6, 3.7, there do not exist $\lceil m/2 \rceil$ completely independent spanning trees in these graphs. By Corollary 3.2, there exist $\lfloor m/2 \rfloor$ completely independent spanning trees in $K_{m-1} \Box C_n$. From these $\lfloor m/2 \rfloor$ completely independent spanning trees in $K_{m-1} \Box C_n$, we can construct $\lfloor m/2 \rfloor$ completely independent spanning trees in $K_m \Box C_n$. The graph $K_m \Box C_n$ contains n vertices $u_0, \ldots, u_{n-1}$ not in $K_{m-1} \Box C_n$, with $u_j \in V(K^j)$ for $j =$

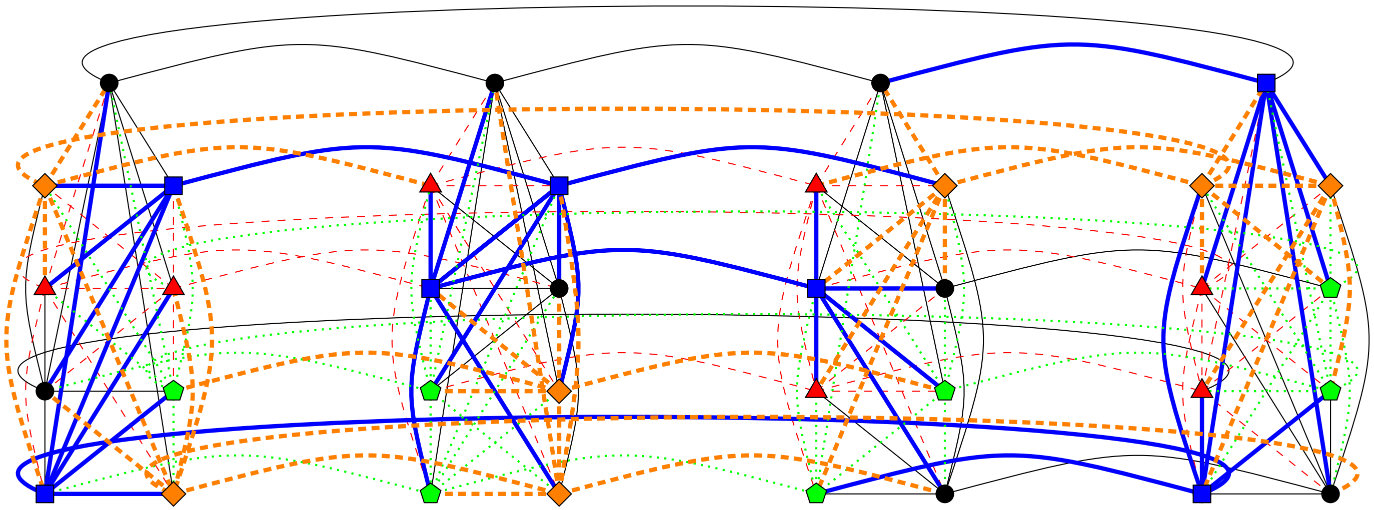

Figure 7: Five completely independent spanning trees in $K_9 \square C_4$.

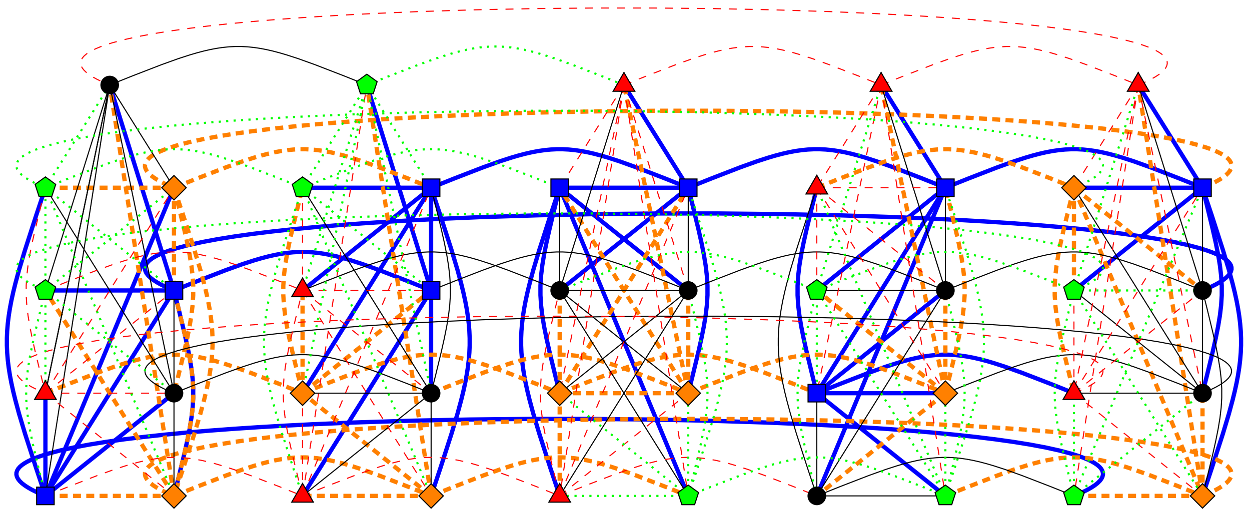

Figure 8: Five completely independent spanning trees in $K_9 \Box C_5$.

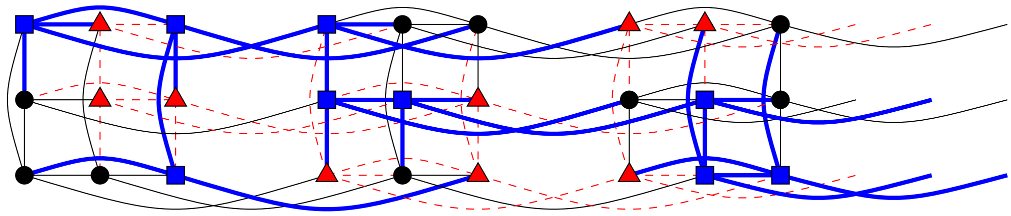

Figure 9: The pattern for the three completely independent spanning trees of $TM(3,3,3q)$, with $q \geq 2$.

$0,\ldots,n-1$. For each $1 \leq i \leq \lfloor m/2 \rfloor$, it suffices to add an edge between u_j, $1 \leq j \leq n$, and a vertex of $V_j(T_i)$ to obtain $\lfloor m/2 \rfloor$ completely independent spanning trees in $K_m \square C_n$. $\square$

4 3-dimensional toroidal grids

Hasunuma and Morisaka [7] have shown that there are two completely independent spanning trees in any 2-dimensional toroidal grid and left as an open problem the question of whether there are n completely independent spanning trees in any n-dimensional toroidal grid, for $n \geq 3$. In this section we give a partial answer for $n = 3$ by finding three completely independent spanning trees in some 3-dimensional toroidal grids.

Let n_1, n_2 and n_3 be positive integers, $3 \leq n_1 \leq n_2 \leq n_3$. The 3-dimensional toroidal grid $TM(n_1,n_2,n_3)$ is the Cartesian product of three cycles: $C_{n_1} \square C_{n_2} \square C_{n_3}$. We let $V(TM(n_1,n_2,n_3)) = \{(i,j,k) | 0 \leq i < n_1, 0 \leq j < n_2, 0 \leq k < n_3\}$ and $E(TM(n_1,n_2,n_3)) = \{(i,j,k)\ (i',j',k') | i \equiv i' \pm 1 \pmod{n_1}, j = j', k = k' \vee i = i', j \equiv j' \pm 1 \pmod{n_2}, k = k' \vee i = i', j = j', k \equiv k' \pm 1 \pmod{n_3}\}$. In the remainder of the section, the integers i, j and k in a vertex (i,j,k) are considered modulo n_1, n_2 and n_3, respectively.

By a *level* of $TM(3,3,q)$ we mean a subgraph of it induced by the vertices with the same third coordinate.

Proposition 4.1. *Let p, p' and q be positive integers such that $gcd(p,p',q) = 1$. There exist three completely independent spanning trees in $TM(3p,3p',3q)$.*

Proof. We define three completely independent spanning trees T_1, T_2 and T_3 in $TM(3p,3p',3q)$ as follows: for $j \in \{0,1,2\}$,
$E(T_{j-1}) = \{(i+j, j-i, i)(1+i+j, -i+j, i), (i+j, j-i, i)(i+j, 1-i+j, i), (i+j, j-i, i)(i+j, j-i, 1+i), (i+j, j-i, i)(i+j, -1-i+j, i), (1+i+j, j-i, i)(2+i+j, j-i, i), (1+i+j, j-i, i)(1+i+j, j-i, 1+i), (1+i+j, j-i, i)(1+i, j-i-1, i), (i+j, 1-i+j, i)(1+i+j, 1-i+j, i), (i+j, 1-i+j, i)(i+j, 1-i+j, 1+i) | i \in \{0,\ldots,pp'q-1\} - (j, j+1, 0)(j, j+1, -1)$.
We require $gcd(p,p',q) = 1$, in order that T_1, T_2, T_3 contain every vertex of

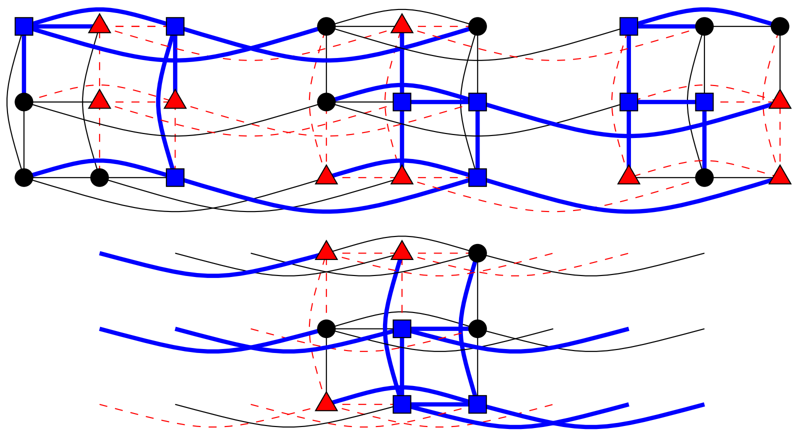

Figure 10: The three completely independent spanning trees on the last four levels of $TM(3,3,q)$, for $q \equiv 1 \pmod 3$ and $q > 2$.

$TM(3p, 3p', 3q)$, i.e. every edge is different for each value of i, $0 \le i \le pp'q - 1$. Figure 9 describes the pattern on three levels for these three spanning trees for $p = 1$ and $p' = 1$.

□

Proposition 4.2. *For any integer $q \ge 3$, there exists three completely independent spanning trees in $TM(3,3,q)$.*

Proof. First, if $q \equiv 0 \pmod 3$, then Proposition 4.1 allows us to conclude. For $q \equiv 1 \pmod 3$ ($q \equiv 2 \pmod 3$, respectively), we define three completely independent spanning trees by using the pattern of Proposition 4.1 for every level except the last four (five, respectively) ones. If $q \equiv 1 \pmod 3$, the trees are completed on the last four levels as depicted in Figure 10 (the corresponding edge sets are given in Appendix B.1). If $q \equiv 2 \pmod 3$, the trees are completed on the last five levels as depicted in Figure 11 (the corresponding edge sets are given in Appendix B.2). □

5 Conclusion

We conclude this paper by listing a few open problems:

1. Determine conditions which ensure that there exist r completely independent spanning trees in a graph.

2. Does any $2r$-connected graph with sufficiently large girth admit r completely independent spanning trees?

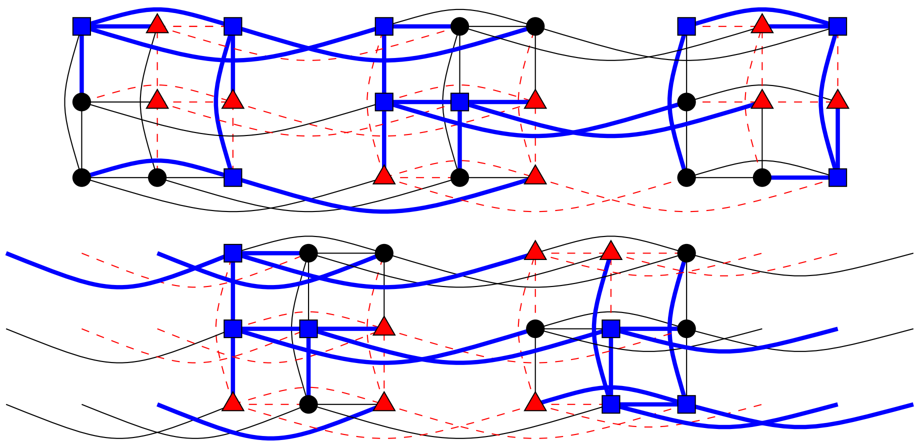

Figure 11: The three completely independent spanning trees on the last five levels of $TM(3,3,q)$, for $q \equiv 2 \pmod 3$ and $q > 2$.

3. Is it true that in every 4-regular graph which is 4-connected, there exist 2 completely independent spanning trees?

4. Does the 6-dimensional hypercube $Q_6 = C_4 \square C_4 \square C_4$ admit 3 completely independent spanning trees?

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A Edge sets of the trees from Section 3

A.1 Three completely independent spanning trees in $K_5 \square C_4$

$E(T_1) = \{u_0^0u_3^0, u_0^0u_2^0, u_3^0u_1^0, u_3^0u_4^0, u_3^1u_1^1, u_3^1u_4^1, u_2^2u_1^2, u_2^2u_4^2, u_0^3u_2^3, u_0^3u_3^3, u_2^3u_1^3, u_2^3u_4^3,$
$u_0^0u_0^1, u_3^0u_3^1, u_2^1u_2^2, u_3^1u_3^2, u_0^2u_0^3, u_2^2u_2^3, u_0^3u_0^4\};$
$E(T_2) = \{u_1^0u_0^0, u_1^0u_2^0, u_0^1u_4^1, u_0^1u_2^1, u_0^1u_3^1, u_4^1u_1^1, u_3^2u_4^2, u_3^2u_1^2, u_3^2u_2^2, u_4^2u_0^2, u_1^3u_3^3, u_1^3u_0^3,$
$u_1^3u_4^3, u_3^3u_2^3, u_4^0u_4^1, u_4^1u_4^2, u_3^2u_3^3, u_1^3u_1^4, u_3^3u_3^4\};$
$E(T_3) = \{u_2^0u_4^0, u_2^0u_3^0, u_4^0u_0^0, u_4^0u_1^0, u_1^1u_2^1, u_1^1u_0^1, u_2^1u_3^1, u_2^1u_4^1, u_0^2u_1^2, u_0^2u_2^2, u_0^2u_3^2, u_1^2u_4^2,$
$u_4^3u_0^3, u_4^3u_3^3, u_2^0u_2^1, u_1^1u_1^2, u_1^2u_1^3, u_2^3u_2^4, u_4^3u_4^4\}.$

A.2 Three completely independent spanning trees in $K_5 \square C_5$

$E(T_1) = \{u_0^0u_3^0, u_0^0u_2^0, u_3^0u_1^0, u_3^0u_4^0, u_0^1u_3^1, u_0^1u_2^1, u_0^1u_4^1, u_3^1u_1^1, u_2^2u_0^2, u_2^2u_1^2, u_2^3u_4^3, u_2^3u_0^3,$
$u_4^3u_1^3, u_4^3u_3^3, u_0^4u_2^4, u_0^4u_1^4, u_2^4u_3^4, u_2^4u_4^4, u_0^0u_0^1, u_3^1u_3^2, u_2^2u_2^3, u_4^2u_4^3, u_2^3u_2^4, u_0^4u_0^5\};$
$E(T_2) = \{u_1^0u_0^0, u_1^0u_2^0, u_4^1u_2^1, u_4^1u_3^1, u_0^2u_4^2, u_0^2u_3^2, u_4^2u_1^2, u_4^2u_2^2, u_0^3u_3^3, u_0^3u_4^3, u_3^3u_1^3, u_3^3u_2^3,$
$u_1^4u_3^4, u_1^4u_2^4, u_1^4u_4^4, u_3^4u_0^4, u_1^0u_1^1, u_4^0u_4^1, u_0^1u_0^2, u_4^1u_4^2, u_0^2u_0^3, u_3^3u_3^4, u_1^4u_1^5, u_3^4u_3^5\};$
$E(T_3) = \{u_2^0u_4^0, u_2^0u_3^0, u_4^0u_0^0, u_4^0u_1^0, u_1^1u_2^1, u_1^1u_0^1, u_1^1u_4^1, u_2^1u_3^1, u_1^2u_3^2, u_1^2u_0^2, u_3^2u_2^2, u_3^2u_4^2,$
$u_1^3u_0^3, u_1^3u_2^3, u_4^4u_0^4, u_4^4u_3^4, u_2^0u_2^1, u_1^1u_1^2, u_1^2u_1^3, u_3^2u_3^3, u_1^3u_1^4, u_4^3u_4^4, u_2^4u_2^5, u_4^4u_4^5\}.$

A.3 Four completely independent spanning trees in $K_7 \square C_3$

$E(T_1) = \{u_0^0u_1^0, u_0^0u_3^0, u_0^0u_5^0, u_0^0u_6^0, u_0^1u_2^1, u_0^1u_4^1, u_0^1u_5^1, u_2^1u_1^1, u_2^1u_3^1, u_2^1u_6^1, u_2^2u_4^2, u_2^2u_5^2,$
$u_2^2u_6^2, u_4^2u_0^2, u_4^2u_1^2, u_4^2u_3^2, u_0^0u_0^1, u_2^1u_2^2, u_2^0u_2^2, u_4^0u_4^2\};$
$E(T_2) = \{u_1^0u_2^0, u_1^0u_3^0, u_1^0u_5^0, u_2^0u_0^0, u_2^0u_4^0, u_2^0u_6^0, u_6^1u_0^1, u_6^1u_3^1, u_6^1u_4^1, u_6^1u_5^1, u_1^2u_0^2, u_1^2u_2^2,$
$u_1^2u_3^2, u_1^2u_6^2, u_6^2u_4^2, u_6^2u_5^2, u_1^0u_1^1, u_2^0u_2^1, u_6^1u_6^2, u_1^0u_1^2\};$
$E(T_3) = \{u_3^0u_2^0, u_3^0u_4^0, u_3^0u_5^0, u_4^0u_0^0, u_4^0u_1^0, u_4^0u_6^0, u_1^1u_0^1, u_1^1u_3^1, u_1^1u_4^1, u_1^1u_6^1, u_4^1u_2^1, u_4^1u_5^1,$
$u_3^2u_0^2, u_3^2u_2^2, u_3^2u_5^2, u_3^2u_6^2, u_4^0u_4^1, u_1^1u_1^2, u_4^1u_4^2, u_3^0u_3^2\};$
$E(T_4) = \{u_5^0u_2^0, u_5^0u_4^0, u_5^0u_6^0, u_6^0u_1^0, u_6^0u_3^0, u_3^1u_0^1, u_3^1u_4^1, u_3^1u_5^1, u_5^1u_1^1, u_5^1u_2^1, u_0^2u_2^2, u_0^2u_5^2,$
$u_0^2u_6^2, u_5^2u_1^2, u_5^2u_4^2, u_5^0u_5^1, u_6^0u_6^1, u_3^1u_3^2, u_5^1u_5^2, u_0^0u_0^2\}.$

A.4 Four completely independent spanning trees in $K_7 \square C_4$

$E(T_1) = \{u_0^0u_1^0, u_0^0u_3^0, u_0^0u_5^0, u_0^0u_6^0, u_0^1u_1^1, u_0^1u_1^1, u_0^1u_2^1, u_0^1u_3^1, u_2^1u_5^1, u_2^1u_6^1, u_2^2u_3^2, u_2^2u_5^2, u_2^2u_6^2,$
$u_5^2u_0^2, u_5^2u_1^2, u_5^2u_4^2, u_4^3u_3^3, u_4^3u_5^3, u_4^3u_6^3, u_5^3u_0^3, u_5^3u_1^3, u_0^0u_0^1, u_2^0u_2^1, u_2^1u_2^2, u_2^2u_2^3, u_5^2u_5^3, u_4^0u_4^3\}$
$E(T_2) = \{u_1^0u_2^0, u_1^0u_4^0, u_1^0u_5^0, u_2^0u_0^0, u_2^0u_3^0, u_2^0u_6^0, u_1^1u_2^1, u_1^1u_4^1, u_1^1u_6^1, u_6^1u_0^1, u_6^1u_3^1, u_6^1u_5^1, u_4^2u_2^2,$
$u_4^2u_3^2, u_4^2u_6^2, u_6^2u_0^2, u_6^2u_5^2, u_2^3u_0^3, u_2^3u_3^3, u_2^3u_4^3, u_2^3u_5^3, u_1^0u_1^1, u_1^1u_1^2, u_6^1u_6^2, u_6^2u_6^3, u_1^0u_1^3, u_2^0u_2^3\}$
$E(T_3) = \{u_3^0u_4^0, u_3^0u_1^0, u_3^0u_5^0, u_4^0u_0^0, u_4^0u_2^0, u_4^0u_6^0, u_4^1u_2^1, u_4^1u_5^1, u_4^1u_6^1, u_5^1u_0^1, u_5^1u_1^1, u_1^2u_0^2, u_1^2u_2^2,$
$u_1^2u_4^2, u_1^2u_6^2, u_1^3u_0^3, u_1^3u_2^3, u_1^3u_3^3, u_1^3u_4^3, u_3^3u_5^3, u_3^3u_7^3, u_3^0u_3^1, u_4^0u_4^1, u_5^1u_5^2, u_1^2u_1^3, u_3^2u_3^3, u_3^0u_3^3\}$
$E(T_4) = \{u_5^0u_2^0, u_5^0u_4^0, u_5^0u_6^0, u_6^0u_1^0, u_6^0u_3^0, u_3^1u_1^1, u_3^1u_2^1, u_3^1u_4^1, u_3^1u_5^1, u_0^2u_2^2, u_0^2u_3^2, u_0^2u_4^2, u_3^2u_1^2,$
$u_3^2u_5^2, u_3^2u_6^2, u_0^3u_3^3, u_0^3u_4^3, u_0^3u_6^3, u_6^3u_1^3, u_6^3u_2^3, u_6^3u_5^3, u_6^0u_6^1, u_0^1u_0^2, u_3^1u_3^2, u_0^2u_0^3, u_0^0u_0^3, u_6^0u_6^3\}.$

A.5 Five completely independent spanning trees in $K_9 \square C_4$

$E(T_1) = \{u_0^0u_2^0, u_0^0u_4^0, u_0^0u_5^0, u_0^0u_8^0, u_5^0u_1^0, u_5^0u_3^0, u_5^0u_6^0, u_5^0u_7^0, u_0^1u_2^1, u_0^1u_4^1, u_0^1u_6^1, u_0^1u_7^1, u_4^1u_1^1,$
$u_4^1u_3^1, u_4^1u_5^1, u_4^1u_8^1, u_0^2u_3^2, u_0^2u_4^2, u_0^2u_6^2, u_4^2u_1^2, u_4^2u_8^2, u_8^2u_2^2, u_8^2u_5^2, u_8^2u_7^2, u_8^3u_1^3, u_8^3u_2^3, u_8^3u_3^3, u_8^3u_6^3,$
$u_8^3u_7^3, u_0^0u_0^1, u_0^1u_0^2, u_4^2u_4^3, u_8^2u_8^3, u_0^0u_0^3, u_5^0u_5^3\};$

$E(T_2) = \{u_3^0u_0^0, u_3^0u_4^0, u_3^0u_7^0, u_3^0u_8^0, u_4^0u_1^0, u_4^0u_2^0, u_4^0u_5^0, u_4^0u_6^0, u_1^1u_0^1, u_1^1u_2^1, u_1^1u_6^1, u_1^1u_7^1, u_1^1u_8^1,$
$u_1^2u_0^2, u_1^2u_2^2, u_1^2u_5^2, u_1^2u_7^2, u_1^2u_8^2, u_5^2u_4^2, u_5^2u_6^2, u_3^3u_2^3, u_3^3u_5^3, u_3^3u_6^3, u_3^3u_7^3, u_5^3u_0^3, u_5^3u_1^3, u_5^3u_4^3, u_5^3u_8^3,$
$u_3^0u_3^1, u_4^0u_4^1, u_1^1u_1^2, u_5^1u_5^2, u_3^2u_3^3, u_5^2u_5^3, u_3^0u_3^3\};$
$E(T_3) = \{u_2^0u_1^0, u_2^0u_3^0, u_2^0u_5^0, u_2^0u_7^0, u_7^0u_0^0, u_7^0u_4^0, u_7^0u_6^0, u_7^0u_8^0, u_2^1u_3^1, u_2^1u_4^1, u_2^1u_5^1, u_2^1u_6^1, u_3^1u_0^1,$
$u_3^1u_1^1, u_3^1u_7^1, u_3^1u_8^1, u_3^2u_1^2, u_3^2u_4^2, u_3^2u_5^2, u_3^2u_6^2, u_6^2u_8^2, u_0^3u_2^3, u_0^3u_3^3, u_0^3u_4^3, u_0^3u_7^3, u_0^3u_8^3, u_7^3u_1^3, u_7^3u_5^3,$
$u_7^3u_6^3, u_2^0u_2^1, u_2^1u_2^2, u_3^1u_3^2, u_0^2u_0^3, u_7^2u_7^3, u_7^0u_7^3\};$
$E(T_4) = \{u_6^0u_0^0, u_6^0u_1^0, u_6^0u_2^0, u_6^0u_3^0, u_6^0u_8^0, u_5^1u_0^1, u_5^1u_1^1, u_5^1u_3^1, u_5^1u_7^1, u_5^1u_8^1, u_7^1u_2^1, u_7^1u_4^1, u_7^1u_6^1,$
$u_6^2u_1^2, u_6^2u_2^2, u_6^2u_4^2, u_6^2u_7^2, u_6^2u_8^2, u_7^2u_0^2, u_7^2u_3^2, u_7^2u_5^2, u_4^3u_2^3, u_4^3u_3^3, u_4^3u_6^3, u_4^3u_7^3, u_4^3u_8^3, u_6^3u_0^3, u_6^3u_1^3,$
$u_6^3u_5^3, u_5^0u_5^1, u_7^0u_7^1, u_7^1u_7^2, u_6^2u_6^3, u_4^0u_4^3, u_6^0u_6^3\};$
$E(T_5) = \{u_1^0u_0^0, u_1^0u_3^0, u_1^0u_7^0, u_1^0u_8^0, u_8^0u_2^0, u_8^0u_4^0, u_8^0u_5^0, u_6^1u_3^1, u_6^1u_4^1, u_6^1u_5^1, u_6^1u_8^1, u_8^1u_0^1, u_8^1u_2^1,$
$u_8^1u_7^1, u_2^2u_0^2, u_2^2u_3^2, u_2^2u_4^2, u_2^2u_5^2, u_2^2u_7^2, u_1^3u_0^3, u_1^3u_2^3, u_1^3u_3^3, u_1^3u_4^3, u_2^3u_5^3, u_2^3u_6^3, u_2^3u_7^3, u_1^0u_1^1, u_6^0u_6^1,$
$u_8^0u_8^1, u_6^1u_6^2, u_8^1u_8^2, u_1^2u_1^3, u_2^2u_2^3, u_1^0u_1^3, u_8^0u_8^3\}.$

A.6 Five completely independent spanning trees in $K_9 \square C_4$

$E(T_1) = \{u_0^0u_2^0, u_0^0u_3^0, u_0^0u_5^0, u_0^0u_6^0, u_0^0u_7^0, u_6^0u_1^0, u_6^0u_4^0, u_6^0u_8^0, u_6^1u_1^1, u_6^1u_2^1, u_6^1u_5^1, u_6^1u_7^1, u_6^1u_8^1,$
$u_3^2u_0^2, u_3^2u_1^2, u_3^2u_4^2, u_3^2u_6^2, u_3^2u_8^2, u_4^2u_2^2, u_4^2u_5^2, u_4^2u_7^2, u_4^3u_0^3, u_4^3u_2^3, u_4^3u_3^3, u_4^3u_7^3, u_7^3u_1^3, u_7^3u_5^3, u_7^3u_8^3,$
$u_4^4u_0^4, u_4^4u_2^4, u_4^4u_6^4, u_4^4u_8^4, u_6^4u_1^4, u_6^4u_3^4, u_6^4u_5^4, u_0^0u_0^1, u_6^0u_6^1, u_3^1u_3^2, u_4^1u_4^2, u_4^2u_4^3, u_4^3u_4^4, u_6^3u_6^4, u_7^3u_7^4, u_6^0u_6^4\};$
$E(T_2) = \{u_7^0u_1^0, u_7^0u_2^0, u_7^0u_4^0, u_7^0u_6^0, u_7^0u_8^0, u_3^1u_1^1, u_3^1u_4^1, u_3^1u_6^1, u_3^1u_7^1, u_3^1u_8^1, u_7^1u_0^1, u_7^1u_2^1, u_7^1u_6^1,$
$u_0^2u_1^2, u_0^2u_4^2, u_0^2u_5^2, u_0^2u_7^2, u_0^2u_8^2, u_7^2u_2^2, u_7^2u_3^2, u_7^2u_6^2, u_0^3u_1^3, u_0^3u_5^3, u_0^3u_8^3, u_1^3u_2^3, u_1^3u_3^3, u_1^3u_4^3, u_1^3u_6^3,$
$u_0^4u_1^4, u_0^4u_5^4, u_0^4u_6^4, u_0^4u_7^4, u_5^4u_2^4, u_5^4u_3^4, u_5^4u_4^4, u_5^4u_8^4, u_3^0u_3^1, u_7^0u_7^1, u_7^1u_7^2, u_0^2u_0^3, u_7^2u_7^3, u_0^3u_0^4, u_0^0u_0^4, u_5^0u_5^4\};$
$E(T_3) = \{u_4^0u_0^0, u_4^0u_3^0, u_4^0u_7^0, u_4^0u_8^0, u_7^0u_1^0, u_7^0u_2^0, u_7^0u_5^0, u_7^0u_6^0, u_2^1u_1^1, u_2^1u_3^1, u_2^1u_4^1, u_2^1u_5^1, u_2^1u_8^1,$
$u_4^1u_0^1, u_4^1u_6^1, u_4^1u_7^1, u_1^2u_2^2, u_1^2u_4^2, u_1^2u_5^2, u_1^2u_7^2, u_1^2u_8^2, u_2^2u_0^2, u_2^2u_3^2, u_2^2u_6^2, u_2^3u_0^3, u_2^3u_3^3, u_2^3u_5^3, u_2^3u_7^3,$
$u_5^3u_1^3, u_5^3u_4^3, u_5^3u_6^3, u_5^3u_8^3, u_2^4u_0^4, u_2^4u_1^4, u_2^4u_3^4, u_2^4u_6^4, u_2^4u_8^4, u_4^0u_4^1, u_2^1u_2^2, u_2^2u_2^3, u_2^3u_2^4, u_5^3u_5^4, u_4^0u_4^4, u_7^0u_7^4\};$
$E(T_4) = \{u_1^0u_0^0, u_1^0u_3^0, u_1^0u_4^0, u_1^0u_8^0, u_3^0u_2^0, u_3^0u_5^0, u_3^0u_6^0, u_3^0u_7^0, u_0^1u_1^1, u_0^1u_2^1, u_0^1u_3^1, u_0^1u_5^1, u_0^1u_6^1,$
$u_1^1u_4^1, u_1^1u_7^1, u_1^1u_8^1, u_8^2u_2^2, u_8^2u_4^2, u_8^2u_5^2, u_8^2u_6^2, u_8^2u_7^2, u_3^3u_0^3, u_3^3u_5^3, u_3^3u_7^3, u_3^3u_8^3, u_8^3u_1^3, u_8^3u_2^3, u_8^3u_4^3,$
$u_8^3u_6^3, u_3^4u_0^4, u_3^4u_4^4, u_3^4u_7^4, u_3^4u_8^4, u_7^4u_2^4, u_7^4u_5^4, u_7^4u_6^4, u_1^0u_1^1, u_0^1u_0^2, u_1^1u_1^2, u_3^2u_3^3, u_8^2u_8^3, u_3^3u_3^4, u_1^0u_1^4, u_3^0u_3^4\};$
$E(T_5) = \{u_2^0u_1^0, u_2^0u_4^0, u_2^0u_6^0, u_2^0u_8^0, u_8^0u_0^0, u_8^0u_3^0, u_8^0u_7^0, u_5^1u_1^1, u_5^1u_3^1, u_5^1u_4^1, u_5^1u_8^1, u_8^1u_0^1, u_8^1u_7^1,$
$u_5^2u_2^2, u_5^2u_3^2, u_5^2u_6^2, u_5^2u_7^2, u_6^2u_0^2, u_6^2u_1^2, u_6^2u_4^2, u_6^3u_0^3, u_6^3u_2^3, u_6^3u_3^3, u_6^3u_4^3, u_6^3u_7^3, u_1^4u_3^4, u_1^4u_4^4, u_1^4u_5^4,$
$u_1^4u_8^4, u_8^4u_0^4, u_8^4u_6^4, u_8^4u_7^4, u_2^0u_2^1, u_5^0u_5^1, u_8^0u_8^1, u_5^1u_5^2, u_6^1u_6^2, u_8^1u_8^2, u_5^2u_5^3, u_6^2u_6^3, u_1^3u_1^4, u_8^3u_8^4, u_2^0u_2^4, u_8^0u_8^4\}.$

B Edge sets of the trees from Section 4

B.1 Three completely independent spanning trees in the last four levels of $TM(3,3,q)$

$E(T_1) = \{(0,0,0)(1,0,0), (0,0,0)(0,1,0), (0,0,0)(0,2,0), (0,0,0)(0,0,1),$
$(1,0,0)(1,2,0), (1,0,0)(2,0,0), (1,0,0)(1,0,1), (0,1,0)(1,1,0), (0,1,0)(0,1,1),$
$(0,1,1)(1,1,1), (0,1,1)(0,2,1), (0,1,1)(0,1,2), (0,2,1)(1,2,1), (0,2,1)(2,2,1),$
$(0,2,1)(0,2,2), (2,2,1)(2,0,1), (2,2,1)(2,1,1), (1,0,2)(0,0,2), (1,0,2)(2,0,2),$
$(1,0,2)(1,2,2), (1,0,2)(1,0,3), (1,2,2)(2,2,2), (1,2,2)(1,1,2), (1,2,2)(1,2,3),$
$(2,2,2)(2,1,2), (2,2,2)(2,2,3), (0,1,3)(1,1,3), (0,1,3)(0,0,3), (0,1,3)(2,1,3),$
$(0,1,3)(0,1,4), (2,1,3)(2,0,3), (2,1,3)(2,2,3), (2,1,3)(2,1,4), (2,2,3)(0,2,3),$
$(2,2,3)(2,2,4)\};$
$E(T_2) = \{(1,1,0)(2,1,0), (1,1,0)(1,0,0), (1,1,0)(1,2,0), (1,1,0)(1,1,1),$

$(2,1,0)(0,1,0), (2,1,0)(2,0,0), (2,1,0)(2,1,1), (1,2,0)(2,2,0), (1,2,0)(1,2,1),$
$(0,0,1)(1,0,1), (0,0,1)(0,1,1), (0,0,1)(0,2,1), (1,0,1)(2,0,1), (1,0,1)(1,2,1),$
$(1,0,1)(1,0,2), (1,2,1)(2,2,1), (1,2,1)(1,2,2), (0,0,2)(2,0,2), (0,0,2)(0,2,2),$
$(0,0,2)(0,0,3), (2,0,2)(2,1,2), (2,0,2)(2,2,2), (2,0,2)(2,0,3), (2,1,2)(0,1,2),$
$(2,1,2)(1,1,2), (2,1,2)(2,1,3), (0,0,3)(1,0,3), (0,0,3)(0,2,3), (0,0,3)(0,0,4),$
$(0,2,3)(1,2,3), (0,2,3)(0,1,3), (0,2,3)(0,2,4), (1,2,3)(2,2,3), (1,2,3)(1,1,3),$
$(1,2,3)(1,2,4)\}$;
$E(T_3) = \{(2,0,0)(0,0,0), (2,0,0)(2,2,0), (2,0,0)(2,0,1), (0,2,0)(1,2,0),$
$(0,2,0)(2,2,0), (0,2,0)(0,1,0), (0,2,0)(0,2,1), (2,2,0)(2,1,0), (2,2,0)(2,2,1),$
$(2,0,1)(0,0,1), (2,0,1)(2,1,1), (2,0,1)(2,0,2), (1,1,1)(2,1,1), (1,1,1)(1,0,1),$
$(1,1,1)(1,2,1), (2,1,1)(0,1,1), (2,1,1)(2,1,2), (0,1,2)(1,1,2), (0,1,2)(0,0,2),$
$(0,1,2)(0,2,2), (0,1,2)(0,1,3), (1,1,2)(1,0,2), (1,1,2)(1,1,3), (0,2,2)(1,2,2),$
$(0,2,2)(2,2,2), (0,2,2)(0,2,3), (1,0,3)(2,0,3), (1,0,3)(1,1,3), (1,0,3)(1,2,3),$
$(1,0,3)(1,0,4), (2,0,3)(0,0,3), (2,0,3)(2,2,3), (2,0,3)(2,0,4), (1,1,3)(2,1,3),$
$(1,1,3)(1,1,4)\}$.

B.2 Three completely independent spanning trees in the last five levels of $TM(3,3,q)$

$E(T_1) = \{(0,0,0)(1,0,0), (0,0,0)(0,1,0), (0,0,0)(0,2,0), (0,0,0)(0,0,1),$
$(1,0,0)(1,2,0), (1,0,0)(2,0,0), (1,0,0)(1,0,1), (0,1,0)(1,1,0), (0,1,0)(0,1,1),$
$(1,0,1)(2,0,1), (1,0,1)(1,2,1), (1,2,1)(2,2,1), (1,2,1)(1,1,1), (1,2,1)(1,2,2),$
$(2,2,1)(0,2,1), (2,2,1)(2,1,1), (2,2,1)(2,2,2), (0,0,2)(1,0,2), (0,0,2)(2,0,2),$
$(0,0,2)(0,1,2), (0,0,2), (0,0,3), (1,0,2)(1,1,2), (1,0,2)(1,0,3), (0,1,2)(2,1,2),$
$(0,1,2)(0,2,2), (0,1,2)(0,1,3), (1,0,3)(2,0,3), (1,0,3)(1,2,3), (1,0,3)(1,0,4),$
$(1,2,3)(2,2,3), (1,2,3)(1,1,3), (1,2,3)(1,2,4), (2,2,3)(0,2,3), (2,2,3)(2,1,3),$
$(2,2,3)(2,2,4), (0,1,4)(1,1,4), (0,1,4)(0,0,4), (0,1,4)(2,1,4), (0,1,4)(0,1,5),$
$(2,1,4)(2,0,4), (2,1,4)(2,2,4), (2,1,4)(2,1,5), (2,2,4)(0,2,4), (2,2,4)(2,2,5)\}$;
$E(T_2) = \{(1,1,0)(2,1,0), (1,1,0)(1,0,0), (1,1,0)(1,2,0), (1,1,0)(1,1,1),$
$(2,1,0)(0,1,0), (2,1,0)(2,0,0), (2,1,0)(2,1,1), (1,2,0)(2,2,0), (1,2,0)(1,2,1),$
$(0,0,1)(1,0,1), (0,0,1)(2,0,1), (0,0,1)(0,2,1), (0,0,1)(0,0,2), (2,0,1)(2,1,1),$
$(2,0,1)(2,2,1), (2,0,1)(2,0,2), (2,1,1)(0,1,1), (1,1,2)(0,1,2), (1,1,2)(2,1,2),$
$(1,1,2)(1,2,2), (1,1,2)(1,1,3), (2,1,2)(2,2,2), (2,1,2)(2,1,3), (1,2,2)(0,2,2),$
$(1,2,2)(1,0,2), (1,2,2)(1,2,3), (0,0,3)(1,0,3), (0,0,3)(2,0,3), (0,0,3)(0,2,3),$
$(0,0,3)(0,0,4), (2,0,3)(2,1,3), (2,0,3)(2,2,3), (2,0,3)(2,0,4), (2,1,3)(0,1,3),$
$(2,1,3)(2,1,4), (0,0,4)(1,0,4), (0,0,4)(0,2,4), (0,0,4)(0,0,5), (0,2,4)(1,2,4),$
$(0,2,4)(0,1,4), (0,2,4)(0,2,5), (1,2,4)(2,2,4), (1,2,4)(1,1,4), (1,2,4)(1,2,5)\}$;
$E(T_3) = \{(2,0,0)(0,0,0), (2,0,0)(2,2,0), (2,0,0)(2,0,1), (0,2,0)(1,2,0),$
$(0,2,0)(2,2,0), (0,2,0)(0,1,0), (0,2,0)(0,2,1), (2,2,0)(2,1,0), (2,2,0)(2,2,1),$
$(0,1,1)(1,1,1), (0,1,1)(0,0,1), (0,1,1)(0,2,1), (0,1,1)(0,1,2), (1,1,1)(2,1,1),$
$(1,1,1)(1,0,1), (1,1,1)(1,1,2), (0,2,1)(1,2,1), (2,0,2)(1,0,2), (2,0,2)(2,1,2),$
$(2,0,2)(2,2,2), (2,0,2)(2,0,3), (0,2,2)(2,2,2), (0,2,2)(0,0,2), (0,2,2)(0,2,3),$
$(2,2,2)(1,2,2), (2,2,2)(2,2,3), (0,1,3)(1,1,3), (0,1,3)(0,0,3), (0,1,3)(0,2,3),$
$(0,1,3)(0,1,4), (1,1,3)(2,1,3), (1,1,3)(1,0,3), (1,1,3)(1,1,4), (0,2,3)(1,2,3),$
$(0,2,3)(0,2,4), (1,0,4)(2,0,4), (1,0,4)(1,1,4), (1,0,4)(1,2,4), (1,0,4)(1,0,5),$
$(2,0,4)(0,0,4), (2,0,4)(2,2,4), (2,0,4)(2,0,5), (1,1,4)(2,1,4), (1,1,4)(1,1,5)\}$.